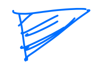
Curves, surfaces up to Gauss

① Finishing turning tangents thm:

(1) 

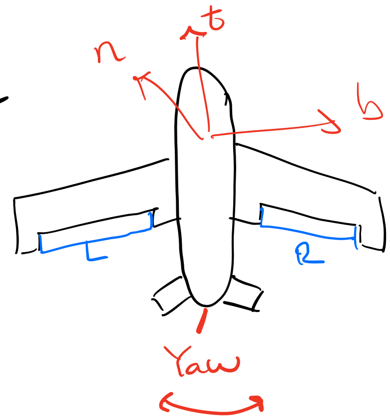
(2) $d\Theta = \eta\, d\theta + d\int 1-\eta$

$\Rightarrow$ degree = # of preimages w/ sign.

(Also (lifting picture))

Q: When are two arclength param curves $\alpha, \beta : I \to \mathbb{R}^3$ the same up to rigid motions?

A: Use an airplane without Yaw



Pitch: ~~jets~~ $\Delta_L + \Delta_R$

Roll: $\Delta_L - \Delta_R$

Minor technicality: If pitch $= 0$, roll doesn't change path.
Simplest sol'n: Assume pitch $\neq 0$; wlog pitch > 0.

Frenet Frame (1847)

Defn If $\alpha : I \to \mathbb{R}^3$ is a curve (param by arclength), define

$$t(s) = \alpha'(s) \quad \text{tangent}$$

$$\kappa(s) = |t'(s)| = |\alpha''(s)| \quad \text{curvature} \quad \text{(compare signed curv)}$$

Assuming $\kappa \neq 0$, define

$$n(s) = \frac{\alpha''(s)}{\kappa(s)} \quad \text{normal} \quad (\text{check } n \perp t)$$

$$b(s) = t \wedge n \quad \text{binormal} \quad \text{(cross product)}$$

$F = [t \ n \ b]$ is the Frenet frame of α

Defn span(t, n) is the osculating plane.

Can think of $F : I \to SO(3)$ $\qquad F^T F = id.$

Write $F' = \dfrac{dF}{dt} \in T_F SO(3)$

$$(F^T)'F + F^T F' = 0 \iff (F^{-1}F')^T + F^{-1}F' = 0$$

$\Rightarrow F^{-1}F'$ is skew-symmetric

Recall Lie algebra of $SO(3) \leftrightarrow T_e SO(3)$ = skew-sym
$\leftrightarrow$ left-invt vector fields

$\Rightarrow \exists$ canonical map $dL_{F^{-1}} : T_F SO(3) \to T_e SO(3)$

For matrices $dL_{F^{-1}}(F') = \underbrace{F^{-1}F'}_{\substack{\text{as matrices} \\ \text{b/c } F^{-1} \text{ linear}}} \rightsquigarrow F^{-1}dF$ Maurer-Cartan form.

$F^{-1}F'$:

$$[t \ \ n \ \ b]' = [t \ \ n \ \ b]\begin{bmatrix} 0 & -x & -y \\ x & 0 & -z \\ y & z & 0 \end{bmatrix}$$

Since the Frenet Frame is orthonormal, $F^TF = I$,
$F^{-1}F'$ is skew-symmetric

$$\Rightarrow [t' \ \ n' \ \ b'] = [t \ \ n \ \ b]\begin{bmatrix} 0 & -\square & \square \\ \square & 0 & \square \\ -\square & -\square & 0 \end{bmatrix}$$

Because $t' \cdot n = \kappa$ and $t' \cdot b = 0$, this matrix is actually of the form

$$\begin{bmatrix} 0 & -\kappa & 0 \\ \kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix}$$

sign convention do Carmo

where τ is a new property, called torsion.

Dfn The torsion is $b' \cdot n$.

Rmk: pitch is κ and roll is τ.

Fund. Thm. of space curves:

① $\forall$ $\kappa > 0$, τ on I, $\exists$ $\alpha : I \to \mathbb{R}^3$ w/ curvature κ, torsion τ

② Such α is unique up to rigid motions

Pf ① Is actually way harder than in 2D b/c [tnb]

need not commute w/ $\begin{bmatrix} 0 & -\kappa & \\ \kappa & & \tau \\ & -\tau & \end{bmatrix}$, so you can't nec diagonalize

Need Fund. Thm. of linear ODE ← Aside: $F' = FA$

$\Rightarrow F' = e^{M} F_0$

completeness then $\Longrightarrow$ exists for all time.

$\alpha = \int t$

② Linearity If $F, \tilde{F}$ solve same eqn, define M by $F(t_0) = M\hat{F}(t_0)$

$$F' = F \, A = \begin{bmatrix} 0 & -\kappa & \\ \kappa & & \tau \\ & -\tau & \end{bmatrix}$$

$$\tilde{F}' = \tilde{F} A \qquad \tilde{F} = MF$$

$$M'F + MF' = MFA \Rightarrow M'F = 0 \Rightarrow M = \text{const}$$

$\Rightarrow$ differ by left multiplication by (orthogonal) M

Then
$$t = F\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$
$$\tilde{t} = MF\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = Mt$$

Integrate again $\rightsquigarrow$ $\tilde{\alpha} = M\alpha + \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}$

$\lrcorner$

Now, if $\alpha : I \to S \subseteq \mathbb{R}^3$ is an arclength-parametrized curve on an oriented surface S, the <u>Darboux frame</u> of α is the orthonormal framing $[T \;\; V \;\; N]$ where

$$T(s) = \alpha'(s)$$

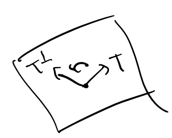
$$N(s) = N_{\alpha(s)}$$

$$V(s) = N(s) \wedge T(s)$$

(See Exercise 4-4.14)

<u>Rmks</u>:

- Don't need to assume $\kappa \neq 0$
- (Exercise) A more natural definition is
    - $T(s)$ is the unit tangent to trace(α) determined by the orientation of α
    - $V(s)$ is $T(s)^{\perp}$ in the tangent space of S at $\alpha(s)$, where the meaning of $\perp$ is determined by the orientation of S



- $N(s)$ is $T \wedge V$ (where the convention for $\wedge$ is determined by the orientation of $\mathbb{R}^3$!)

Since $[T\ V\ N]$ is orthonormal, we still have

$$[T'\ V'\ N'] = [T\ V\ N] \begin{bmatrix} 0 & -a & -b \\ a & 0 & -c \\ b & c & 0 \end{bmatrix}$$

for some functions $a(s), b(s), c(s)$, (see lectures 4.1 and 4.2)

which are now invariants up to rigid motion of the pair (α, S).

$\langle T', T\rangle = \frac{1}{2}(|T|^2)' = 0$

Airplane analogy: now we have a rudder

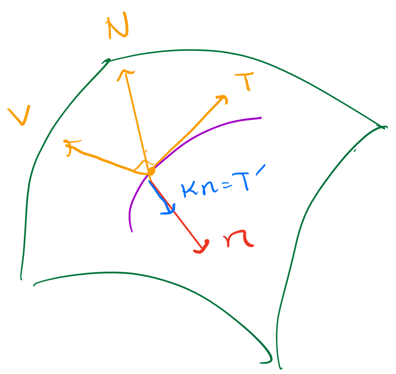
<u>Dfn</u> • $\langle T', N\rangle = b = -\langle N', T\rangle$ is the <u>normal curvature</u> of α in S. Do Carmo calls this k_n.

more on these later

• $\langle T', V\rangle = a = -\langle V', T\rangle$ is the <u>geodesic curvature</u> of α in S.

• $\langle V', N\rangle = c = -\langle N', V\rangle$ is the <u>geodesic torsion</u> of α in S.

<u>Prop</u> $\overset{|T'|}{\underset{\shortparallel}{}}\kappa = \sqrt{a^2+b^2}$

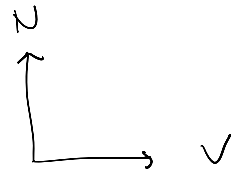


Γ T' is in the plane spanned by V and N (since α is arclength-parametrized)

and V and N are orthonormal, so

$$\kappa = |T'| = \sqrt{\langle T', V\rangle^2 + \langle T', N\rangle^2} = \sqrt{a^2+b^2}$$

The V, N plane:



<u>Dfn</u> ①. If T' is parallel to V, α is an asymptotic curve of S. (i.e. $\langle T', N\rangle = 0$)

much more later ②. If T' is parallel to N, α is a <u>geodesic</u> of S. (i.e. $\langle T', V\rangle = 0$)

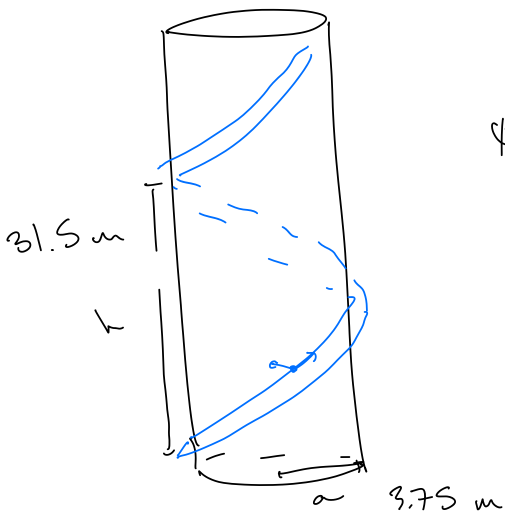
. If N' is parallel to T, then α is a <u>line of curvature</u> of S. (i.e. $\langle N', V\rangle = 0$)

① $\Leftrightarrow$ (T, V, N) Frenet

② $\Leftrightarrow$ $(T, N, -V)$ Frenet

Morgan's problem:

Test tower



cut out of a flat sheet.
What radius?

$$\varphi(u) = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a \cos u \\ a \sin u \\ h \frac{u}{2\pi} \end{pmatrix}$$

$$\frac{d\varphi}{du} = \begin{pmatrix} -a \sin u \\ a \cos u \\ \frac{h}{2\pi} \end{pmatrix} \qquad \frac{d^2\varphi}{du^2} = \begin{pmatrix} -a \cos u \\ -a \sin u \\ 0 \end{pmatrix}$$

$$|\varphi'| = \sqrt{a^2 + \frac{h^2}{2\pi}} =: c$$

$L = 2\pi c$ = length of one section

guess: circle of radius $c \sim 6.26$ m?

too small

Better guess?

$$s = \frac{u}{c}$$

$$\frac{d\varphi}{ds} = \frac{d\varphi}{du} \frac{1}{c}$$

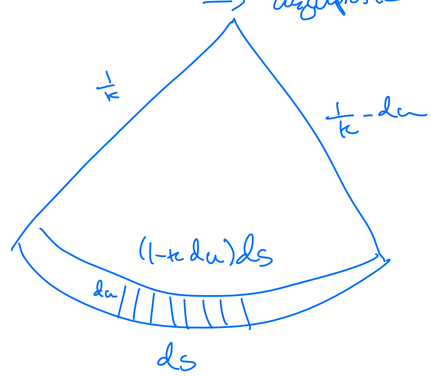
$$\frac{d^2\varphi}{ds^2} = \frac{d^2\varphi}{du^2} \left(\frac{du}{ds}\right)^2$$

$$\kappa = \frac{a}{c^2}$$

$$\frac{1}{\kappa} \sim 10.45 \text{ m}$$

Darboux frame = Frenet frame

→ asymptotic curve



$\frac{1}{\kappa} - du = \frac{1}{\kappa}(1 - \kappa du)$

$ds = \frac{1}{\kappa} d\theta$